

# FLAWS OF CALOGERO-MOSER SYSTEMS

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## 1. INTRODUCTION

The Calogero-Moser (or CM) particle system [Ca1, Ca2] and its generalizations appear, in a variety of ways, in integrable systems, nonlinear PDE, representation theory, and string theory. Moreover, the partially completed CM systems—in which dynamics of particles are continued through collisions—have been identified as meromorphic Hitchin systems (see, for example, [BBT, DM, GN, HM, HN, Kr2, Kr3, N1] among others). This Hitchin-type description gives natural “geometric action-angle variables” for the CM system.

Motivated by relations of the CM system to nonlinear PDE [BN1, BN2], we introduce a new class of generalizations of the spin CM particle systems, the *framed* (rational, trigonometric and elliptic) CM systems. We give two algebro-geometric descriptions of these systems: the first, perhaps more familiar, description uses meromorphic  $GL_n$  Hitchin systems with decorations (framing data) on (cuspidal, nodal and smooth) cubic curves. The second description identifies these phase spaces with moduli spaces of one-dimensional sheaves on corresponding “twisted” ruled surfaces.

We also present a simple geometric formulation of the flows of all meromorphic  $GL_n$  Hitchin systems (with no regularity assumptions) as *tweaking* flows on spectral sheaves. Using this formulation, we show that all spin and framed CM systems are identified with hierarchies of tweaking flows on the corresponding spectral sheaves. This generalizes the well-known description of spinless CM systems in terms of tangential covers (see [TV1, TV2]). In [BN2], we prove that a Fourier transform identifies the framed CM systems (in their spectral incarnation) as the particle systems describing the motion of poles of all meromorphic (rational, trigonometric, and elliptic) solutions of (generalized) multicomponent KP hierarchies.

We begin with an overview of the paper. Sections 2 and 3 review relevant background, Section 4 introduces framed CM systems and the corresponding spectral sheaves on ruled surfaces, and Section 5 contains the description of flows of Hitchin and framed CM systems.

**1.1. Review of Calogero-Moser Systems.** In Section 2, we review the definition of the Calogero-Moser (CM) particle systems. The Calogero-Moser systems are a family of completely integrable classical hamiltonian systems, describing particles on the line interacting with a quadratic potential. The systems come in three basic variants: rational, trigonometric and elliptic, according to the type of function used in the potential. We will work with complexified Calogero-Moser systems, in which the three variants naturally correspond to systems of interacting particles on a one-dimensional complex group  $\mathbf{G}$ , namely  $\mathbf{C}$ ,  $\mathbf{C}^\times$  or an elliptic curve. The Calogero-Moser systems also have natural partial completions, in which we allow

the positions of particles to collide [KKS, W1]. The (completed) Calogero-Moser systems feature in a variety of seemingly unrelated areas, in particular as describing the motion of poles of meromorphic solutions to KdV and KP equations (see [AMM, CC, Kr1, Kr2, TV2, W1] and [Be, GW, W2] for surveys), and as analogs of Hilbert schemes in noncommutative geometry [BW, BGK1, KKO, BrNe].

The Calogero-Moser systems have a natural generalization, in which the particles carry spins, which are covectors and vectors  $u_i, v_i$  in an auxiliary vector space  $\mathbf{C}^k$  [GH]. The dynamics of the spin CM systems depends on the spins only through the pairings  $f_{ij} = u_i(v_j)$ , so one usually considers instead a reduced version (the *Euler-Calogero-Moser system*) in which we only keep track of the  $f_{ij}$  [BBKT, R, N1]. From the point of view of noncommutative geometry and KP equations (e.g. [BGK2, BN2]), it is the full phase space of the spin CM hierarchy, and its framed generalizations presented in this paper, which play a central role. We review the definition of the completed rational and trigonometric spin CM systems in terms of quiver data (pairs of matrices, vectors and covectors satisfying a shifted moment map condition) in Section 2.2.

**1.2. Calogero-Moser and Hitchin Systems.** We begin Section 3 by reviewing (in Section 3.1) the basic setup, the family of Weierstrass cubic curves  $E$  (irreducible genus one curves). These come in three variants, according to the type of the smooth locus  $\mathbf{G} \subset E$ : rational (the cuspidal cubic, with  $\mathbf{G} = \mathbf{C}$ ), trigonometric (the nodal cubic, with  $\mathbf{G} = \mathbf{C}^\times$ ), and elliptic ( $\mathbf{G} = E$  is an elliptic curve).

In Section 3.3 we review the relation between meromorphic Hitchin systems [Hi, M, DM] on cubic curves and Euler-Calogero-Moser systems (following [N1, GN] as well as the reviews [DM, BBT]). The Hitchin formulation is a variant of the Lax form with spectral parameter for elliptic CM systems, due to Krichever [Kr2] (see also [Kr3] where elliptic CM systems appear as part of a general hamiltonian theory of Lax operators on algebraic curves).

Recall that the  $(GL_n)$  Hitchin systems are integrable systems on moduli spaces of vector bundles on a curve, equipped with a Higgs field (endomorphism-valued one-form). A key ingredient in this approach to particle systems is the identification ([FM1], see Section 3.2) between positions of particles (configurations of points, or more generally torsion sheaves, on  $\mathbf{G}$ ) and vector bundles on  $E$ , which is a special case of the Fourier-Mukai transform on cubic curves [BuK, BN2].

**1.3. Framed Calogero-Moser Systems and Spectral Sheaves.** Section 4 introduces and studies the framed Calogero-Moser systems. These systems are best described geometrically using a nontrivial affine bundle (the Serre surface)  $E^\natural \rightarrow E$  modelled on  $T^*E$ . This formulation for the elliptic (spinless) Calogero-Moser system was explored in detail by Treibich-Verdier in their work on tangential covers [TV1, TV2], and in [DW, D] (see [DM]). One of our goals is to generalize this description to include the rational and trigonometric cases and to CM particles with spins or *framings*.

The ruled surface  $\overline{E}^\natural \rightarrow E$  over a cubic curve  $E$  is discussed in Section 4.1. It comes equipped with a section  $E_\infty$ , whose complementary affine bundle  $E^\natural = \overline{E}^\natural \setminus E_\infty$  is the natural home for the Weierstrass  $\zeta$ -function. The surface  $E^\natural$  is naturally birational to  $T^*E$  (again using the  $\zeta$ -function), so that the complement of the fibers over the basepoint  $b \in E$  (identity element of the group  $\mathbf{G}$ ) are identified.

In Section 4.2 we introduce the notion of twisted Higgs field, which relate to sheaves on  $E^\natural$  in the same way that ordinary Higgs fields relate to sheaves on  $T^*E$ . Using the birational identification of  $E^\natural$  and  $T^*E$  we have a simple bijection between regular and twisted (framed or unframed) Higgs fields, which shifts the Higgs field by  $\zeta \text{Id}$ . This shift provides a geometric origin (the transition from  $T^*E$  to  $E^\natural$ ) for the appearance of the shift by  $\text{Id}$  in the hamiltonian reduction and Hitchin system descriptions of CM hamiltonians (and corresponds, under the extended Fourier-Mukai transform [BN2], to a transition from sheaves on  $T^*E$  to  $\mathcal{D}$ -modules).

For any torsion coherent sheaf  $T$  on the group  $\mathbf{G} \subset E$ , we introduce in Section 4.2.2 the *T-framed Calogero-Moser systems*, which are Hitchin systems on *T-framed* Higgs (or twisted Higgs) bundles. The framing consists of a factorization of the polar parts of a Higgs field into a map to  $T$ . This generalizes the usual  $k$ -spin Calogero-Moser systems, which are recovered when the framing  $T = \mathbf{C}^k$ , considered as a sum of skyscrapers  $\mathcal{O}_b^{\oplus k}$  at the basepoint of  $\mathbf{G}$ . The framing is also analogous to that appearing in the definition of Nakajima quiver varieties [Na]. Thanks to the identification between vector bundles on  $E$  and configurations of particles, these systems have the form of generalized spin particle systems.

In Section 4.3 we introduce the spaces of *framed CM spectral sheaves*, which provide a geometric phase space for spin (and framed) CM systems. Recall that the phase spaces of Hitchin systems on a curve  $X$  are described geometrically as spaces of spectral sheaves on the cotangent bundle  $T^*X$ —i.e. line bundles (or more generally torsion-free sheaves) supported on curves in the surface  $T^*X$ . In fact the description of an integrable system by spectral sheaves is a geometric version of transforming the system in action-angle variables—the support curve is invariant under the system and plays the role of the action variables, while the line bundle on the curve plays the role of the angle variable.

For CM systems, the natural spectral sheaves live on the ruled surface  $\overline{E}^\natural$ . Given a torsion sheaf  $T$  on the smooth locus  $\mathbf{G} \subset E$ , we define a *T-framed CM spectral sheaf* on  $E$  to be a torsion-free sheaf supported on a curve in  $\overline{E}^\natural$ , whose restriction to  $E_\infty$  is identified with  $T$ —see Definition 4.6 for a precise definition (the algebraic geometry of related linear series on  $\overline{E}^\natural$  is studied in [T]). The torsion sheaf  $T$  again plays the role of the spin variables of the corresponding Calogero-Moser particles—in particular we'll show (Corollary 4.12) that the  $k$ -spin CM system is realized as the case where  $T$  is the vector space  $\mathbf{C}^k$ , considered as a skyscraper sheaf  $\mathcal{O}_b^{\oplus k}$  at the basepoint  $b \in E$ .

In Section 4.4 we identify moduli spaces of  $T$ -framed CM spectral sheaves and Higgs bundles:

**Theorem 1.1** (Theorem 4.11). *There is a canonical isomorphism  $\mathfrak{CM}_n(E, T) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E, T)$  between the moduli of  $T$ -framed spectral sheaves and  $T$ -framed Higgs fields.*

The identification is based on a result of Katzarkov, Orlov and Pantev [KOP] which uses Koszul duality to identify moduli spaces of framed sheaves on ruled surfaces over curves in terms of linear algebra data on the curve. It turns out that the Koszul data for CM spectral sheaves are precisely framed twisted Higgs bundles, so that the Higgs field gives the structure of the underlying spectral sheaf while the factorization of the poles into spin variables gives the framing of the

spectral sheaf. In particular, when  $T = \mathcal{O}_b^k$ , we find that the phase spaces of the rational, trigonometric and elliptic spin CM particles are identified with spaces of framed spectral sheaves on  $E^\natural$ .

**1.4. Flows on Spectral Sheaves.** In Section 5, the heart of the paper, we study flows on moduli spaces of spectral sheaves. We first discuss in Section 5.1 the general principle (tweaking), whereby sheaves are deformed by an infinitesimal version of tensoring with line bundles. Explicitly we can construct deformations of arbitrary sheaves from germs of meromorphic functions. Namely, multiplication by such a function gives a germ of a meromorphic endomorphism of any sheaf, using which we “change the transition functions” (or define an  $\text{Ext}^1$  class).

In Section 5.2 we present a geometric description of the flows of meromorphic  $GL_n$  Hitchin systems on an algebraic curve  $X$  as tweaking flows (Theorem 5.4). (Related geometric pictures with various restrictions on the allowed spectral curves appear for example in [AB, AHP, DM, BF1, LM1, LM2].) Namely we point out an obvious bijection between Hitchin hamiltonians for vector bundles on a curve  $X$  and classes in  $H^1(T^*X, \mathcal{O})$ , and likewise for meromorphic Hitchin systems and meromorphic germs on  $T^*X$ . It is then an easy check that the corresponding tweaking and hamiltonian flows agree (this is a global analog of the trivial spectral description of the hamiltonian flows of trace polynomials on  $T^*GL_n$ ). We thus realize the Hitchin flows not in terms of the action of line bundles on each specific spectral curve but uniformly as the infinitesimal version of the action of line bundles on  $T^*X$  (i.e. as the action of a commutative Lie *algebra*, rather than Lie algebroid).

**Theorem 1.2** (Theorem 5.4). *The hamiltonian flows on the moduli of (possibly meromorphic) Higgs bundles on a curve  $X$  are given by the multiplication action of classes in  $H^1(T^*X, \mathcal{O})$  (or generally meromorphic germs on  $T^*X$ ) on the corresponding spectral sheaves.*

This concrete realization of Hitchin flows has various applications. As an example we describe, as Corollary 5.6, a simple generalization of the Compatibility Theorem of Donagi-Markman [DM] and Li-Mulase [LM2] relating Hitchin and Heisenberg (or KdV) flows, dropping all assumptions on the regularity of the Higgs field.

Finally, in Section 5.4 we define the framed Calogero-Moser hierarchies, as tweaking flows on framed spectral sheaves. Specifically, we tweak CM spectral sheaves by principal parts of functions with poles along the curve  $E_\infty$  at infinity. We are then able to identify explicitly all of these flows with Hitchin hamiltonian flows, and in particular with the explicit form of the spin (and framed) Calogero-Moser hamiltonians on particles. We summarize as follows:

**Theorem 1.3** (Theorem 5.10).

- (1) *The flows of the rational, trigonometric and elliptic spin CM hamiltonian systems are identified with explicit tweaking flows (see Definition 5.9) along  $E_\infty$  on  $(\mathcal{O}_b^k\text{-framed})$  CM spectral sheaves on  $E^\natural$ .*
- (2) *More generally, for any torsion sheaf  $T$  on  $\mathbf{G} \subset E$  with  $E$  a cubic curve, the flows of the  $T$ -framed Calogero-Moser hamiltonians are identified (under the isomorphism  $\mathfrak{CM}_n(E, T) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E, T)$  above) with explicit tweaking flows (Definition 5.9) along  $E_\infty$  on  $T$ -framed CM spectral sheaves.*
- (3) *For simple framing  $T$  (for example in the spinless case  $T = \mathcal{O}_b$ ), the CM hamiltonian flows generate all tweaking flows along  $E_\infty$ .*

**1.5. Motivation.** The impetus for the present work comes from the correspondence between meromorphic solutions of soliton equations of KP type and particle systems of CM type. In [BN2] (see [BN1] for an overview) we establish a very broad form of this correspondence, generalizing and refining (in particular) the results of [Kr2, W1, TV1] in the spinless case. Namely, we apply an extension of the Fourier-Mukai transform to the spaces of framed CM spectral sheaves studied in this paper, extending the identification of the underlying vector bundles with configurations of particles. We obtain an identification of these phase spaces with moduli spaces of *framed  $\mathcal{D}$ -bundles* on  $E$ . These moduli spaces are noncommutative analogs of Hilbert schemes (in the rank one case) or framed torsion-free sheaves on  $T^*E$ , and the isomorphism may be considered a separation of variables à la Sklyanin (see [GNR]) for elliptic Hitchin systems. This generalizes the relation between rational Calogero-Moser spaces and ideals in the Weyl algebra [LB, BW, BGK1]. Moreover, we show that framed  $\mathcal{D}$ -bundles provide a natural geometric phase space for the meromorphic (rational, trigonometric and elliptic) multicomponent KP hierarchy, and that the isomorphism of moduli spaces identifies the KP and CM flows. The positions of the CM particles are identified with the “singularities” of the  $\mathcal{D}$ -bundles, which are the poles of the corresponding meromorphic KP solution. Thus, framed CM systems describe the motion of poles of general meromorphic solutions of multicomponent KP hierarchies.

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## 2. REVIEW OF CALOGERO-MOSER SYSTEMS

**2.1. Introducing the Spin Calogero-Moser System.** In this section we discuss the spin Calogero-Moser system, following [GH, BBKT, N1, R]—see also the chapter in [BBT]. See [BN1] for a review of the usual (spinless) complexified Calogero-Moser system following [KKS, W1, N1, GN, ER, Ma].

Connected one-dimensional complex algebraic groups  $\mathbf{G}$  fall into three classes: the additive group  $\mathbf{C}$ , the multiplicative group  $\mathbf{C}^\times$ , and the one-parameter family of elliptic curves  $E$ . These cases fall under the monikers *rational*, *trigonometric* and *elliptic* according to the type of functions on the universal cover  $\mathbf{C}$  which correspond to meromorphic functions on  $\mathbf{G}$ .

The  $k$ -spin  $n$ -particle Calogero-Moser system is a hamiltonian system describing  $n$  identical particles on  $\mathbf{G}$  equipped with spins in the auxiliary  $k$ -dimensional vector space  $\mathbf{C}^k$ . Thus, consider  $n$  distinct points (positions)  $q_1, \dots, q_n$  in  $\mathbf{G}$ , momenta  $p_i \in \mathbf{C}$  and spin vectors and covectors  $v_i \in \mathbf{C}^k$ ,  $u_i \in (\mathbf{C}^k)^*$  ( $1 \leq i \leq n$ ), all up to the simultaneous action of the symmetric group  $S_n$ . Let  $f_{ij} = u_i(v_j) \in \mathbf{C}$  ( $i \neq j$ )

be the contraction of the  $i$ th covector with the  $j$ th vector. The hamiltonian for the spin Calogero-Moser system is given by

$$H = H_2 = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i < j} f_{ij} f_{ji} U(q_i - q_j).$$

Here the potential function  $U$  has a single second order pole at the origin of  $\mathbf{G}$ : in terms of coordinates on the complex line (the universal cover of  $\mathbf{G}$ ),  $U$  has one of the forms

<b>Rational:</b>	$U(q) = \frac{1}{q^2},$
<b>Trigonometric:</b>	$U(q) = \frac{1}{\sin^2(q)},$
<b>Elliptic:</b>	$U(q) = \wp(q)$

where  $\wp(q)$  is the Weierstrass  $\wp$ -function attached to the elliptic curve  $E$ . The *spinless* case  $k = 1$ ,  $f_{ij} = 1$  is the classical Calogero-Moser particle system.

The spin Calogero-Moser hamiltonian depends only on the contractions  $f_{ij} = u_i(v_j)$  (as do the higher integrals of motion discussed below). Thus the dynamics of the system descend to the phase space of the *Euler-Calogero-Moser system* (we follow the terminology of [BBKT]), in which we only keep track of the  $p_i, q_i$  and the matrix  $F = (f_{ij}) \in \mathfrak{gl}_n$  (considered up to the addition of diagonal matrices). This latter system is often referred to as the spin Calogero-Moser system, since its dynamics come from those of the full spin CM system. However, the phase spaces for the two systems are quite different (especially so when  $k > n$ ) and it is the full spin CM phase space that plays a role in the correspondence with the multicomponent KP hierarchy [BN2].

**2.2. CM Matrices.** We briefly recall the description of rational and trigonometric spin Calogero-Moser systems in terms of matrices (or quivers) following [KKS, W1].

Consider the cotangent bundles  $T^*\mathfrak{gl}_n^{r,ss}$  (rational case) and  $T^*GL_n^{r,ss}$  (trigonometric case) of the regular semisimple loci in the Lie algebra and group. These cotangent bundles are identified with the sets of pairs of matrices  $(X, Y)$  with  $X$  having  $n$  distinct eigenvalues ( $q_i \in \mathbf{C}$  in the rational/Lie algebra case,  $q_i \in \mathbf{C}^\times$  in the trigonometric/Lie group case). We now pass to the quotient by the simultaneous conjugation action of  $GL_n$ , which is identified with the phase space of the rational (respectively trigonometric) Euler-Calogero-Moser system, with  $q_i$  the positions of the particles. The corresponding momenta  $p_i$  are recovered as the diagonal entries of  $Y$  in the gauge where  $X$  is diagonal. Finally we have moment maps for the action of  $GL_n$ :

$$\begin{aligned} T^*\mathfrak{gl}_n^{r,ss} &\rightarrow \mathfrak{gl}_n, & X, Y &\mapsto F = [X, Y], \\ T^*GL_n^{r,ss} &\rightarrow \mathfrak{gl}_n, & X, Y &\mapsto F = X^{-1}YX - Y, \end{aligned}$$

giving the spin coordinates  $f_{ij}$  as the off-diagonal entries of the matrix  $F \in \mathfrak{gl}_n$  (or as  $X$ -rescaled versions of the off-diagonal entries of  $Y$ ). Note that this phase space is Poisson, with symplectic leaves labeled by coadjoint orbits. The Calogero-Moser hamiltonian is given by the  $GL_n$ -invariant function  $H_2^{CM} = \frac{1}{2} \text{tr } Y^2$ . The hamiltonians  $H_i^{CM} = \frac{1}{i} \text{tr } Y^i$  ( $i = 1, 2, \dots$ ) are in involution, and define a degenerately integrable hamiltonian system [R].

We may now drop the assumption that the matrix  $X$  is regular semisimple, obtaining a partially completed phase space for the rational and trigonometric Euler-Calogero-Moser systems in which the positions  $q_i$  are allowed to coincide.

**Definition 2.1.** The rational and trigonometric Euler-Calogero-Moser spaces<sup>1</sup> are the quotients  $\mathfrak{ECM}_n(\mathbf{C}) = T^*\mathfrak{gl}_n/GL_n$ ,  $\mathfrak{ECM}_n(\mathbf{C}^\times) = T^*GL_n/GL_n$ . The Calogero-Moser hamiltonians on these spaces are the reductions of the invariant polynomials  $H_i^{CM} = \frac{1}{i} \text{tr } Y^i$ .

In order to describe the rational and trigonometric *spin* Calogero-Moser systems, we consider in addition to the matrices  $X, Y$  also maps  $u : \mathbf{C}^k \rightarrow \mathbf{C}^n$ ,  $v : \mathbf{C}^n \rightarrow \mathbf{C}^k$ . When  $X$  is regular semisimple, we may write  $u, v$  in the basis of  $X$ -eigenvectors, giving the data of  $n$  vectors  $v_i \in \mathbf{C}^k$  and  $n$  covectors  $u_i \in (\mathbf{C}^k)^*$  that are the spin parameters for the  $n$  particles with positions  $\{q_i\}$ . Noting that the variety of quadruples  $(X, Y, u, v)$  is the cotangent bundle of the variety of pairs  $(X, u)$ , we obtain the following definition:

**Definition 2.2.** The rational and trigonometric spin Calogero-Moser spaces are the hamiltonian reductions

$$\mathfrak{CM}_n^k(\mathbf{C}) = T^*(\mathfrak{gl}_n \times \text{Hom}(\mathbf{C}^k, \mathbf{C}^n)) //_{\text{Id}} GL_n,$$

$$\mathfrak{CM}_n^k(\mathbf{C}^\times) = T^*(GL_n \times \text{Hom}(\mathbf{C}^k, \mathbf{C}^n)) //_{\text{Id}} GL_n$$

at the coadjoint orbit  $\{\text{Id}\} \in \mathfrak{gl}_n$ . In other words, these are the varieties of quadruples  $\{X, Y, u, v\}$  with  $[X, Y] + u(v) = \text{Id}$  and  $X^{-1}YX - Y + u(v) = \text{Id}$ , respectively, modulo the simultaneous action of  $GL_n$ . The Calogero-Moser hamiltonians on these spaces are the reductions of the invariant polynomials  $H_i^{CM} = \frac{1}{i} \text{tr } Y^i$ .

*Remark 2.3.* It is clear from the above description that  $\mathfrak{CM}_n^k(\mathbf{C})$  is a (framed) quiver variety [Na] associated to the quiver with one vertex and one loop (the matrix  $X$ ): the vector  $v$  is the framing datum, and  $Y, u$  come from doubling the resulting quiver.

**2.3. Formulas for Rational CM Matrices.** When  $k = 1$ , the spin Calogero-Moser system reduces to the usual spinless Calogero-Moser system, which is the symplectic leaf of the Euler-Calogero-Moser system corresponding to the minimal coadjoint orbit  $\mathbb{O} = \{\text{Id} - u(v)\} \subset \mathfrak{gl}_n$  that consists of traceless matrices of the form “Id minus a rank one matrix.” In other words, the (spinless) rational CM space is

$$\mathfrak{CM}_n = \{(X, Y) \in T^*\mathfrak{gl}_n \mid [X, Y] \in \mathbb{O}\} / GL_n.$$

It is proven in [W1] that this space is a smooth, irreducible affine variety of dimension  $2n$ . It is convenient to realize  $\mathbb{O}$  as the orbit of the matrix

$$(2.1) \quad \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

<sup>1</sup>Here and elsewhere in the paper, we will use the word “spaces” (or “moduli spaces”) in a slightly abusive way. However, all statements in the paper apply equally well to the moduli stack and to any reasonable moduli spaces/varieties that result, so the reader may substitute his or her preferred type of moduli object.

On the open subset where  $X$  has distinct eigenvalues, we may then write coordinates  $(q_i, p_i)$  on (a finite cover of)  $\mathfrak{CM}_n$ :

$$X = \begin{pmatrix} q_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & q_3 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_n \end{pmatrix}, \quad Y = \begin{pmatrix} p_1 & \frac{1}{q_1 - q_2} & \frac{1}{q_1 - q_3} & \cdots & \frac{1}{q_1 - q_n} \\ \frac{1}{q_2 - q_1} & p_2 & \frac{1}{q_2 - q_3} & \cdots & \frac{1}{q_2 - q_n} \\ \frac{1}{q_3 - q_1} & \frac{1}{q_3 - q_2} & p_3 & \cdots & \frac{1}{q_3 - q_n} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ \frac{1}{q_n - q_1} & \frac{1}{q_n - q_2} & \frac{1}{q_n - q_3} & \cdots & p_n \end{pmatrix}.$$

It is easy to see that the hamiltonian  $H = H_2$  in these coordinates recovers the rational Calogero-Moser hamiltonian above. Thus,  $\mathfrak{CM}_n$  provides a completion of the phase space of the rational Calogero-Moser system in which we allow the points  $q_i$  (the eigenvalues of  $X$ ) to collide. In the general spin case, we obtain coordinates  $p_i, q_i$  as diagonal entries just as before and coordinates  $f_{ij}$  ( $i \neq j$ ) with  $Y_{ij} = \frac{f_{ij}}{q_i - q_j}$ .

### 3. CALOGERO-MOSER AND HITCHIN SYSTEMS

**3.1. Cubic Curves.** The classification of 1-dimensional complex algebraic groups is paralleled by the classification of Weierstrass cubic curves, that is, irreducible, reduced complex projective curves  $E$  of arithmetic genus 1:

- **Elliptic:**  $E$  is a smooth elliptic curve (in particular a group), and may be described by an equation of the form  $y^2 = x^3 + ax + b$  where  $4a^3 + 27b^2 \neq 0$ .
- **Trigonometric:**  $E$  is a nodal cubic, and is isomorphic to the curve  $y^2 = x^2(x - 1)$ . Its normalization  $\mathbf{P}^1 \rightarrow E$  identifies two points  $0$  and  $\infty$  to a node on  $E$ , and defines a group structure  $\mathbf{C}^\times = \mathbf{G} \subset E$  on the smooth locus.
- **Rational:**  $E$  is a cuspidal cubic, and is isomorphic to the curve  $y^2 = x^3$ . Its normalization  $\mathbf{P}^1 \rightarrow E$  collapses  $2 \cdot \infty$  to a cusp on  $E$ , and defines a group structure  $\mathbf{C} = \mathbf{G} \subset E$  on the smooth locus.

We will denote the identity element of each group  $\mathbf{G}$  by  $b$ , and the singular point (cusp or node) by  $\infty$ . In all three cases, the smooth locus  $\mathbf{G}$  is identified (as a group) with the Jacobian  $\text{Pic}^0(E)$  via the map  $q \mapsto \mathcal{L}_q = \mathcal{O}(q - b)$ .  $E$  itself is identified with its compactified Jacobian, the moduli space of torsion free sheaves of rank 1 and degree 0 on  $E$ . The singular point corresponds to the unique rank 1, degree 0 torsion-free sheaf that is not locally free, namely the modification  $\mathfrak{m}_\infty(b)$  of the ideal sheaf of  $\infty$ .

The group variety  $\mathbf{G}$  acts on  $E$ , defining the unique nonzero invariant vector field  $\partial$  on  $E$  up to a scalar. Writing the singular cubics in terms of their normalization  $\mathbf{P}^1$ , a choice of  $\partial$  is represented by  $\frac{\partial}{\partial z}$  in the cuspidal case (vanishing to order 2 at  $\infty \in \mathbf{P}^1$ ) and by  $z \frac{\partial}{\partial z}$  in the nodal case (vanishing to order one at  $0, \infty$ ). We will abuse notation to denote the sheaf  $\mathcal{O}_E \cdot \partial$  by  $\mathcal{T}_E$  (note in the nodal case this is the log tangent bundle of  $E$ ). Similarly the dual sheaf will be denoted by  $\Omega_E$ . Both sheaves are trivial line bundles on  $E$ . The total space of  $\Omega_E$  (which is isomorphic to  $E \times \mathbf{C}$ ) will be denoted  $T^*E$ .

**3.2. From Particles to Vector Bundles.** We would like to encode the positions  $q_i \in \mathbf{G}$  of the Calogero-Moser particles in a “Fourier dual” fashion. Recall that  $\mathbf{G}$  is identified with the Jacobian  $\text{Pic}^0(E)$  of the corresponding cubic curve  $E$ , via the map  $q \mapsto \mathcal{L}_q = \mathcal{O}(q - b)$ . Thus,  $n$  distinct points in  $\mathbf{G}$  define a rank  $n$  vector bundle



$W = \bigoplus \mathcal{O}(q_i - b)$  on  $E$  which is semistable of degree zero. Conversely, a generic degree 0 semistable bundle on  $E$  is of the form  $W = \bigoplus \mathcal{O}(q_i - b)$  for  $n$  distinct points  $q_i \in \mathbf{G} \subset E$  (determined up to permutation).

On an elliptic curve  $E$ , we may extend this correspondence as follows. The Fourier-Mukai transform [Muk] identifies semistable degree zero vector bundles  $W$  with degree (length)  $n$  torsion coherent sheaves  $W^\vee$  on  $E$ , in such a way that  $W = \bigoplus \mathcal{O}(q_i - b)$  is identified with the sum of skyscrapers  $\bigoplus \mathcal{O}_{q_i}$ . We may therefore consider the moduli space of such  $W$  as a completion of the configuration space of  $n$  points  $q_i \in E$ .

A similar argument for singular cubic curves identifies certain semistable degree zero vector bundles with configurations of points in the *smooth* locus  $\mathbf{G} \subset E$ :

**Lemma 3.1** ([FM1], Lemma 1.2.5). *Suppose  $W$  is a semistable vector bundle of degree 0 on a singular Weierstrass cubic  $E$ , and that  $\mathbf{P}^1 \simeq \tilde{E} \xrightarrow{n} E$  is the normalization. Then the following are equivalent:*

- (1)  $W$  is identified by the Fourier-Mukai transform with a torsion coherent sheaf  $W^\vee$  supported on  $\mathbf{G} \subset E$ .
- (2)  $n^*W$  is a trivial vector bundle.
- (3)  $W$  has a filtration whose subquotients are line bundles of degree 0 on  $E$ .

For any cubic curve  $E$ , we let  $Bun_n^{ss}(E)$  denote the moduli stack of rank  $n$  semistable vector bundles of degree zero on a cubic curve  $E$ ; for singular  $E$ , we impose in addition any of the equivalent (and open) conditions of Lemma 3.1. Note that we do not at any point pass to the moduli *space* of ( $S$ -equivalence classes of) semistable vector bundles.

**3.3. Calogero-Moser and Hitchin Systems.** In this section we review the identification of the completed Euler-Calogero-Moser systems with meromorphic Hitchin systems on cubic curves following [N1] (see also [GN, ER, BBT]).

We denote by  $Bun_n^{ss}(E, b) \rightarrow Bun_n^{ss}(E)$  the principal  $GL_n$ -bundle parametrizing bundles in  $Bun_n^{ss}(E)$  equipped with a trivialization of the fiber at the identity. The cotangent fiber  $T^*Bun_n^{ss}(E, b)|_W$  at a bundle  $W$  consists of pairs  $(W, \eta)$ , where  $\eta$  is a meromorphic Higgs field  $\eta \in \Gamma(\text{End } W(b))$  on  $W$  with only a simple pole at  $b$ . The group  $GL_n$  admits a hamiltonian action on  $T^*Bun_n^{ss}(E, b)$  induced from its action on  $Bun_n^{ss}(E, b)$  (by changing the trivialization at  $b$ ). Let  $\mathbf{C}_{k,n} = \text{Hom}(\mathbf{C}^k, \mathbf{C}^n)$ , and identify  $T^*\mathbf{C}_{k,n} = \mathbf{C}_{k,n} \times \mathbf{C}_{n,k}$ .

**Definition 3.2.** The Hitchin-Calogero-Moser space  $\mathfrak{H}\mathcal{CM}_n(E)$  associated to the cubic curve  $E$  is the quotient  $T^*Bun_n^{ss}(E, b)/GL_n$ , i.e. the space of Higgs bundles on  $E$  with simple pole at  $b$ .

**Definition 3.3.** A *framed Higgs bundle* is a quadruple  $(W, \eta, u, v)$  where

- (1)  $(W, \eta) \in \mathfrak{H}\mathcal{CM}_n(E)$  is a Higgs bundle on  $E$  with pole at  $b$ ,
- (2)  $u : \mathbf{C}^k \rightarrow W|_b$  and  $v : W|_b \rightarrow \mathbf{C}^k$  are linear maps,

and we require that  $\text{Res}_b \eta + u(v) = \text{Id}$ .

The moduli space of framed Higgs bundles on  $E$  is denoted  $\mathfrak{H}\mathcal{CM}_n^k(\mathbf{G})$  and is identified with the hamiltonian reduction

$$\mathfrak{H}\mathcal{CM}_n^k(\mathbf{G}) = T^*(Bun_n^{ss}(E, b) \times \mathbf{C}_{k,n}) //_{\text{Id}} GL_n.$$

As we explain below, the Hitchin systems on the spaces  $\mathfrak{H}\mathcal{CM}_n(E)$  give a completion of the Euler-Calogero-Moser systems. The spaces  $\mathfrak{H}\mathcal{CM}_n^k(E)$  of framed

Higgs bundles, on the other hand, will model the spin CM systems, and the map  $\mathfrak{H}\mathfrak{CM}_n^k(E) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E)$  forgetting  $u, v$  corresponds to forgetting the spins and remembering only their contractions  $f_{ij}$ . In particular the Hitchin hamiltonians on  $\mathfrak{H}\mathfrak{CM}_n(E)$  pull back to define the framed Hitchin system on  $\mathfrak{H}\mathfrak{CM}_n^k(E)$ . In Section 4.4 we will relate framed Higgs bundles (and their generalizations) to spectral sheaves equipped with a normalization (framing) on a ruled surface.

*Remark 3.4.* The contraction map  $\mathfrak{H}\mathfrak{CM}_n^k(E) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E)$  forgetting  $(u, v)$  identifies the (spinless) Calogero-Moser space  $\mathfrak{CM}_n(E) = \mathfrak{H}\mathfrak{CM}_n^1(E)$  with the subspace (in fact symplectic leaf) of  $\mathfrak{H}\mathfrak{CM}_n(E)$  consisting of Higgs fields with residue in the coadjoint orbit  $\mathbb{O}$ .

**3.4. Matrices and Hitchin Systems.** The spaces of Calogero-Moser matrices  $\mathfrak{CM}_n(\mathbf{C})$  and  $\mathfrak{CM}_n(\mathbf{C}^\times)$  are readily identified with the rational and trigonometric Hitchin-Calogero-Moser spaces  $\mathfrak{H}\mathfrak{CM}_n(E)$ . Namely, by hypothesis, bundles in  $Bun_n^{ss}(E, b)$  have trivial pullback to the normalization  $\mathbf{P}^1$ , so are completely described by the descent data from  $\mathbf{P}^1$  to  $E$ . This descent datum in the nodal case is the identification of the two fibers over the inverse image of the node, hence  $Bun_n^{ss}(E, b) = \mathrm{GL}_n$ . In the cuspidal case, these two points are infinitesimally nearby, and the descent datum becomes a “connection matrix” identifying these two nearby fibers—thus, we have  $Bun_n^{ss}(E, b) = \mathfrak{gl}_n$ . (See [FM2] for more details.)

In general, fix a cubic curve  $E$  and an invariant differential on it. We restrict to the open locus in  $Bun_n^{ss}(E)$  consisting of vector bundles  $W \simeq \bigoplus \mathcal{L}_{q_i}$ , sums of the line bundles  $\mathcal{L}_{q_i} = \mathcal{O}(q_i - b)$  of degree zero associated to  $n$  distinct points  $q_i \in \mathbf{G}$ . Let  $s_{q_i - q_j}$  denote the unique section meromorphic section of  $\mathcal{L}_{q_i - q_j}$  with only a simple pole at  $b$  (normalized using the differential and trivialization of the fiber) and zero at  $q_i - q_j$ . Then it is easy to see that the Higgs field  $\eta \in \mathrm{End}(W)(b) = \bigoplus \mathcal{L}_{q_i - q_j}(b)$  must have the form

$$(3.1) \quad \begin{pmatrix} p_1 & f_{12}s_{q_1 - q_2} & f_{13}s_{q_1 - q_3} & \cdots & f_{1n}s_{q_1 - q_n} \\ f_{21}s_{q_2 - q_1} & p_2 & f_{23}s_{q_2 - q_3} & \cdots & f_{2n}s_{q_2 - q_n} \\ f_{31}s_{q_3 - q_1} & f_{32}s_{q_3 - q_2} & p_3 & \cdots & f_{3n}s_{q_3 - q_n} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ f_{n1}s_{q_n - q_1} & f_{n2}s_{q_n - q_2} & f_{n3}s_{q_n - q_3} & \cdots & p_n \end{pmatrix}$$

for some  $f_{ij} \in \mathbf{C}$  and where the  $p_i$  are sections of  $\mathcal{O}(b)$  on  $E$ , hence constants. In the rational case we have  $s_{q_i - q_j} = \frac{1}{z} - \frac{1}{q_i - q_j}$ , so that writing  $X = \mathrm{diag}(q_i)$  and  $[X, Y] = (f_{ij})$  we have

$$(3.2) \quad \eta = \left( \frac{[X, Y]}{z} + Y \right) dz,$$

with  $Y$  as in equation 3.1 with  $s_{q_i - q_j}$  replaced by  $\frac{1}{q_i - q_j}$ . Similarly in the trigonometric case we replace  $\frac{1}{q_i - q_j}$  by  $\sin(q_i - q_j)$  and

$$(3.3) \quad \eta = \left( \frac{X^{-1}YX - Y}{z} + Y \right) \frac{dz}{z}.$$

**3.5. CM Hamiltonians.** Finally, we give the rational, trigonometric and elliptic spin Calogero-Moser hamiltonians in the Hitchin system description. Recall that the Hitchin hamiltonians on the moduli space of Higgs bundles are all components of traces of powers of the Higgs field. Among these we wish to single out the spin CM hamiltonians.

**Definition 3.5.** The spin CM hamiltonians on  $\mathfrak{H}\mathfrak{CM}_n(E)$  are the functions

$$H_i : (W, \eta) \mapsto \frac{1}{(i+1)} \text{Res}_b \text{Tr}(\eta + \zeta I)^{i+1}.$$

We see from the explicit form of the Higgs fields in the rational and trigonometric cases (Equations 3.2 and 3.3) that  $\frac{1}{i}H_i(X, Y) = H_i^{CM}(Y) = \frac{1}{i} \text{tr} Y^i$  are indeed the CM hamiltonians.<sup>2</sup> Similarly, in the elliptic case one checks that  $\frac{1}{2}H_2$  is the (quadratic) elliptic CM hamiltonian using the identity  $s_{q_i - q_j} s_{q_j - q_i} = \wp$ .

Summarizing, we have the following statement.

**Proposition 3.6** ([N1]). *The Hitchin-Calogero-Moser systems on cuspidal, nodal, and smooth cubic curves are completions of the rational, trigonometric and elliptic Euler-Calogero-Moser systems, respectively. Moreover, we have isomorphisms  $\mathfrak{ECM}_n(\mathbb{C}) \simeq \mathfrak{H}\mathfrak{CM}_n(\mathbb{C})$  and  $\mathfrak{ECM}_n(\mathbb{C}^\times) \simeq \mathfrak{H}\mathfrak{CM}_n(\mathbb{C}^\times)$  of integrable systems.*

#### 4. FRAMED CALOGERO-MOSER SYSTEMS AND SPECTRAL SHEAVES

**4.1. The Surface  $E^\natural$  and the Weierstrass  $\zeta$ -Function.** In this section, we discuss a surface  $E^\natural$  that is the total space of the unique (up to isomorphism) nontrivial rank one affine bundle over a cubic curve  $E$ .

**4.1.1.  $E^\natural$  for Smooth Cubics.** Fix an elliptic curve  $E$ . Let  $A$  denote the Atiyah bundle on  $E$ , the unique (again, up to isomorphism) nontrivial extension of  $\mathcal{O}_E$  by itself. The algebraic surface  $E^\natural$  is the complement of the section  $E_\infty = \mathbb{P}(\mathcal{O}) \cong E$  of the projectivization  $\overline{E}^\natural = \mathbb{P}(A)$  of the Atiyah bundle:

$$E^\natural = \overline{E}^\natural \setminus E_\infty.$$

The resulting surface  $E^\natural$  is Stein (but not affine algebraic), and isomorphic with the moduli space of line bundles with a holomorphic connection on  $E$  (see [TV1, TV2] for more on the geometry of  $E^\natural$ ).

In classical analytic terms, the surface  $E^\natural$  may be viewed as the receptacle for the Weierstrass  $\zeta$ -function of  $E$ , the unique odd function on the universal cover  $\mathbf{C}$  of  $E$  whose derivative  $\zeta'(z) = -\wp(z)$  is minus the Weierstrass  $\wp$ -function of  $E$ . That is, while  $\zeta$  is not doubly-periodic, it differs from its translates by additive constants, so that it determines a well-defined section of an affine  $\mathbf{C}$ -bundle over  $E$ . This surface is readily identified with  $E^\natural$ . Indeed, recall that the Weierstrass  $\sigma$ -function of  $E$  is the unique section of the line bundle  $\mathcal{O}(b)$  with a simple zero at  $b$ , and that  $\zeta = d \log \sigma(z)$  is its logarithmic derivative. This provides the algebraic definition of  $\zeta$ : it is the section of the affine bundle  $\text{Conn}\mathcal{O}(b)$  of connections on  $\mathcal{O}(b)$  with log pole at  $b$  and residue 1 that corresponds to the unique meromorphic connection annihilating  $\sigma$ .

We may now fix  $E^\natural$  up to unique isomorphism by setting  $E^\natural = \text{Conn}\mathcal{O}(b)$ , a twisted cotangent bundle of  $E$  [BB] (i.e. affine bundle for  $\Omega_E \simeq \mathcal{O}_E$  with compatible symplectic structure). Let  $\mathcal{A}$  denote the pushforward of  $\mathcal{O}_{E^\natural}$  to  $E$ , i.e. the algebra of functions on the fibers of  $E$ . Thus  $A = (\mathcal{A})_{\leq 1}$ , the subsheaf of affine functions on  $E^\natural$ , is isomorphic to the Atiyah bundle, and is canonically an extension of  $\mathcal{T}_E$  (which is isomorphic to  $\mathcal{O}_E$ ) by  $\mathcal{O}_E$ . The sheaf  $A$  is also isomorphic as  $\mathcal{O}_E$ -module

<sup>2</sup>Our normalization of the hamiltonians is chosen to be compatible with the Hitchin hamiltonians in the next section.

to  $\mathcal{D}^1(\mathcal{O}_E(b))$ , the sheaf of differential operators of order at most one acting on the line bundle  $\mathcal{O}(b)$ . Concretely, the sheaf  $A$  lies in between

$$(4.1) \quad \mathcal{O}_E \oplus \mathcal{T}_E(-b) \subset A \subset \mathcal{O}_E \oplus \mathcal{T}_E(b),$$

and  $A$  is generated (in the canonical local coordinate near  $b$ ) by  $\mathcal{O}_E \oplus \mathcal{T}_E(-b)$  and the section  $\partial - \zeta$ .

The meromorphic section  $\zeta(z)$  of  $E^\natural \rightarrow E$  defines a trivialization of the affine bundle  $E^\natural \rightarrow E$  away from  $b$ , and hence a canonical birational identification of the cotangent bundle  $T^*E \simeq E \times \mathbf{C}$  with  $E^\natural$ :

$$T^*E \ni (z, \omega(z)) \mapsto (z, \zeta(z) + \omega(z)) \in E^\natural$$

(see [DM] for a geometric description in terms of elementary modifications). Let  $\underline{k}$  denote the meromorphic function on  $E^\natural$ , with polar divisor the fiber over  $b$ , obtained by composing this identification with the projection onto  $\mathbf{C}$ ,  $\underline{k}(z, \zeta(z) + \omega(z)) = \omega(z)$  (or more canonically  $\omega(z)/dz$ ). Let  $\underline{\zeta}$  denote the Laurent expansion of  $\zeta$  at the origin  $0 \in \mathbf{C}$ , considered as a Laurent series on  $E$  at  $b$ . Then it follows that the function  $\underline{t} = \underline{k} + \pi^* \underline{\zeta}$  on  $E^\natural$  near  $F_b$  is regular along the fiber  $F_b$ , and gives a natural affine coordinate on  $E^\natural$  near  $F_b$  (i.e an affine identification of the formal neighborhood of  $F_b \subset E^\natural$  with  $E \times \mathbf{C}$ ). More generally, pullback by  $\underline{k}$  identifies regular functions on  $E^\natural$  with the ring of functions generated by linear functions  $f$  on  $T^*X$  such that  $f + \pi^* \underline{\zeta}$  is regular near  $F_b$  (as is evident from equation 4.1).

**4.1.2.  $E^\natural$  for Singular Cubics.** The definition and properties of  $E^\natural$  extend naturally to general cubics  $E$ . For any Weierstrass cubic  $E$  we have

$$\mathrm{Ext}_E^1(\mathcal{O}, \mathcal{O}) = H^1(E, \mathcal{O}) \cong \mathbf{C}.$$

So  $E$  has a unique nontrivial extension  $A$  of  $\mathcal{O}_E \simeq \mathcal{T}_E$  by  $\mathcal{O}_E$ , up to isomorphism.<sup>3</sup> We again fix  $A = \mathcal{D}^1(\mathcal{O}_E(b))$  (these are differential operators with symbol in  $\mathcal{T}_E$ ). Let  $\overline{E}^\natural \stackrel{\mathrm{def}}{=} \mathrm{Proj}(\mathrm{Sym}^\bullet A)$  denote the associated ruled surface, and  $p : \overline{E}^\natural \rightarrow E$  denote the projection map.

The quotient map  $A \twoheadrightarrow \mathcal{O}_E$  defines a section  $s : E \rightarrow \overline{E}^\natural$ ; we write  $E_\infty = s(E)$  and refer to it as the *section at infinity*. The surface  $E^\natural \stackrel{\mathrm{def}}{=} \overline{E}^\natural \setminus E_\infty$  is called the *twisted (log) cotangent bundle* of  $E$ ; it is the nontrivial torsor over  $\Omega_E$  given by the nonzero class (up to scale) in  $H^1(\Omega_E)$ .

The notion of  $\zeta$ -function and its relation to  $E^\natural$  similarly extend to the singular cases (again considering  $\zeta$  as a log connection on  $\mathcal{O}(b)$ ). Concretely, in the rational case, we have  $\zeta = 1/z$  and in the trigonometric case  $\zeta = 1/\sin(z)$ .

**4.2. Twisted Higgs Fields and Framed CM Systems.** In this section we introduce the notion of *twisted* Higgs fields, which are a modified version of Higgs fields whose spectral curves naturally live in  $E^\natural$  rather than in the cotangent bundle. The two notions are readily identified, but the translation from twisted Higgs to Higgs adds  $\mathrm{Id}$  to the residue at the basepoint, providing a geometric origin to the appearance of  $\zeta \mathrm{Id}$  in the CM Hamiltonian or equivalently of  $\mathrm{Id}$  in the CM moment condition. We then define the framed CM particle system in its Hitchin system formulation.

Recall (Section 4.1) the  $\mathcal{O}_E$ -algebra  $\mathcal{A}$  of functions on  $E^\natural$  and its subsheaf  $A$  (the Atiyah bundle) of affine functions, which is an extension of  $\mathcal{O}_E$  by  $\mathcal{T}_E \simeq \mathcal{O}_E$ .

<sup>3</sup>Recall that  $\mathcal{T}_E$  is the subsheaf of the tangent sheaf generated by the  $\mathbf{G}$ -action.

The structure of Higgs bundle on a vector bundle  $W$  can be written as an action of  $\mathcal{O}_E \oplus \mathcal{T}_E$  on  $W$  extending the  $\mathcal{O}_E$ -module structure, which makes  $W$  into a sheaf on  $T^*E$ . This has a natural twisted analog, in which we replace the  $\mathcal{O}_E \oplus \mathcal{T}_E$ -action by an action of  $A$ , or equivalently a lifting to a sheaf on  $E^\natural$ :

**Definition 4.1.** A (regular) *twisted Higgs bundle* on  $E$  is a pair  $(W, \tilde{\eta})$  where

- (1)  $W$  is a vector bundle on  $E$  and
- (2)  $\tilde{\eta} : A \otimes_{\mathcal{O}_E} W \rightarrow W$  is a map whose restriction to  $\mathcal{O}_E \subset A$  is the identity map of  $W$ .

The relation between Higgs and twisted Higgs fields is given by the Weierstrass  $\zeta$ -function, which may be considered as a splitting of the extension  $A$  away from  $b$ , or as a section of  $A$  with simple pole at  $b$ . Put another way,  $A$  is identified with the subsheaf of  $\mathcal{O}_E \oplus \mathcal{T}_E(b)$  generated by  $\mathcal{O}_E$ ,  $\mathcal{T}_E(-b)$  and the section  $\partial - \zeta$ . It follows that to give a twisted Higgs field  $\tilde{\eta}$  on  $W$  is equivalent to giving an action of  $\partial$  from  $W$  to  $W(b)$ , which becomes regular after subtracting  $\zeta \text{Id}$ . In coordinate-free language, this is a Higgs field  $\eta = \tilde{\eta} - \zeta \text{Id}$  on  $W$  with simple pole whose residue at  $b$  is the identity endomorphism. Recall (Section 4.1) that we have a canonical birational isomorphism between  $T^*E$  and  $E^\natural$  relative to  $E$ , given by the function  $\underline{k}$ .

**Lemma 4.2.** *Let  $W$  denote a vector bundle on  $E$ . There is a bijection between meromorphic Higgs fields  $\eta : W \rightarrow W(*b)$  and meromorphic twisted Higgs fields  $\tilde{\eta} : A \otimes_{\mathcal{O}_E} W \rightarrow W(*b)$  sending  $\eta$  to  $\tilde{\eta} - \zeta \text{Id}$ . The corresponding spectral sheaves on the surfaces  $T^*E$ ,  $E^\natural$  away from the fiber over  $b$  are identified by the birational isomorphism of the surfaces given by  $\zeta$ .*

**4.2.1. Framed Higgs Fields.** Let  $T$  denote a torsion coherent sheaf on  $E$ , with support  $S \subset \mathbf{G}$  a subscheme of the smooth locus of  $E$  (considered as a divisor on  $\mathbf{G}$ ). For a sheaf  $F$  on  $E$  we use the chosen invariant differential on  $E$  to identify  $F(S)/F$  with the restriction  $F|_S$ .

**Definition 4.3.** A *T-framed twisted Higgs bundle* is a quadruple  $(W, \tilde{\eta}, u, v)$  where

- (1)  $W \in \text{Bun}_n^{ss}(E)$ ,
- (2)  $u : T \rightarrow W|_S$  and  $v : W|_S \rightarrow T$  are maps of coherent sheaves, and
- (3)  $\tilde{\eta} : A \otimes_{\mathcal{O}_E} W \rightarrow W(S)$  is a map whose restriction to  $\mathcal{O}_E \subset A$  is the identity map of  $W$ ;

these data must satisfy the following. The restriction of  $\tilde{\eta}$  to  $S$  factors through a map

$$p.p.(\tilde{\eta}) : W|_S = (A \otimes W / \mathcal{O} \otimes W)|_S \rightarrow W(S)|_S \simeq W|_S$$

which we require to satisfy

$$p.p.(\tilde{\eta}) = u(v).$$

We denote the moduli space of framed twisted Higgs bundles by  $\mathfrak{H}\mathfrak{C}\mathfrak{M}_n(E, T)$ .

**Lemma 4.4.** *There is an isomorphism between the moduli spaces of:*

- (1) *T-framed twisted Higgs bundles  $(W, \tilde{\eta}, u, v)$ , and*
- (2) *quadruples  $(W, \eta, u, v)$  where*
  - (a)  *$(W \in \text{Bun}_n^{ss}(E), \eta : W \rightarrow W(S + b))$  is a meromorphic Higgs bundle and*
  - (b)  *$u : T \rightarrow W|_S$  and  $v : W|_S \rightarrow T$  are maps of coherent sheaves*

whose principal parts satisfy

$$p.p.(\eta) = u(v) + p.p.(\zeta \cdot \text{Id}_W) : W|_S \rightarrow W|_{S+b}.$$

In the case  $T = \mathcal{O}_b^k$ , we recover the notion of framed Higgs bundle from Definition 3.3:

$$\mathfrak{H}\mathfrak{CM}_n^k(E) \simeq \mathfrak{H}\mathfrak{CM}_n(E, \mathcal{O}_b^k).$$

The notion of twisted Higgs bundle thus gives a geometric interpretation (the passage from  $T^*E$  to  $E^\natural$ ) for the appearance of  $\text{Id}$  in the CM residue condition on Higgs fields.

**4.2.2. Framed CM Systems.** We now define hamiltonians on the framed Calogero-Moser phase spaces  $\mathfrak{H}\mathfrak{CM}_n(E, T)$  as certain Hitchin hamiltonians, generalizing Definition 3.5 in the spin case  $T = \mathcal{O}_b^k$ . Let  $S = \coprod_{j=1}^\ell x_j \subset E$  denote the set-theoretic support of  $T$ , consisting of  $\ell$  distinct points on  $X$  (counted without multiplicity). We define hamiltonians  $H_{i,b}$  whenever  $b \in S$  as in the spin case, but also a collection of hamiltonians  $H_{i,x_j}$  for each point  $x_j \in S$ , as follows:

**Definition 4.5.** The framed CM hamiltonians are the functions

$$H_{i,x_j} : \mathfrak{H}\mathfrak{CM}_n(E, T) \rightarrow \mathbf{C}, \quad H_{i,x_j}(W, \eta, u, v) = \frac{1}{i+1} \text{Res}_{x_j} \text{Tr} \eta^{i+1}$$

for  $x_j \neq b$ , together with (when  $b \in S$ )

$$H_{i,b}(W, \eta, u, v) = \frac{1}{i+1} \text{Res}_b \text{Tr}(\eta + \zeta \text{Id})^{i+1}.$$

The framed CM systems may be identified with completed particle systems on the group  $\mathbf{G}$ , as in Section 3.4. Namely, we restrict to the open locus in which the vector bundle  $W = \bigoplus \mathcal{O}(q_i - b)$  is canonically a sum of distinct line bundles (up to permutation). The positions of the corresponding particles are then given by the points  $q_i \in \mathbf{G}$  (given by the Fourier-Mukai transform of  $W$ , as in Section 3.2). The decomposition of  $W$  moreover allows us to decompose  $\eta$  into diagonal and off-diagonal components. We identify the momentum  $p_i$  of the particle  $q_i$  as the constant term of the  $i$ th diagonal component of  $\eta$ . It is then immediate that the hamiltonian  $\frac{1}{2}H_{2,b}$  consists of the kinetic term  $\frac{1}{2} \sum p_i^2$  together with other (potential) terms. The other hamiltonians above define integrals of motion for this system. In the case of *simple framing* (that is, when  $T = \bigoplus x_i$  for distinct  $x_i$ ) they form a maximal family of integrals in involution, defining an algebraically completely integrable system. (This follows immediately from the spectral description of framed CM systems, Theorem 5.10.)

**4.3. CM Spectral Sheaves.** Moduli spaces of spectral sheaves (specifically, of line bundles on curves in a Poisson surface) give a wide class of examples of integrable systems (see e.g. [DM, Hu]). The prototypical example of such a setting is the  $(GL_n)$  Hitchin system on the moduli space  $T^*\text{Bun}_n(X)$  of Higgs bundles on a curve  $X$ , which can be described as a moduli space of torsion-free sheaves on curves in  $T^*X$  finite of degree  $n$  over  $X$ . We will similarly realize the spin CM systems in terms of spectral curves on the twisted cotangent bundle  $E^\natural$  of  $E$ . More precisely, we look at torsion-free sheaves on curves in the projective surfaces

$$\overline{T^*E} = T^*E \cup E_\infty \quad \text{and} \quad \overline{E^\natural} = E^\natural \cup E_\infty,$$

for which we fix the behavior along the curves  $E_\infty \equiv E$  at infinity.

**Definition 4.6.** Fix a coherent torsion sheaf  $T$  on  $\mathbf{G}$ . A  $T$ -framed CM spectral sheaf (respectively, *Hitchin spectral sheaf*) is a pair  $(\mathcal{F}, \phi)$  consisting of a coherent sheaf  $\mathcal{F}$  on  $\overline{E}^\natural$  (respectively,  $\overline{T^*E}$ ) of pure dimension one, together with an identification  $\phi : \mathcal{F}|_{E_\infty} \rightarrow T$ , satisfying the following two normalization conditions:

- (i)  $W = p_*\mathcal{F}(-E_\infty)$  is a semistable vector bundle of degree 0; if  $E$  is singular, we also require that the pullback of  $W$  to the normalization of  $E$  is a trivial vector bundle.
- (ii)  $\deg(p_*\mathcal{F}(kE_\infty)) = (k+1)\deg(T)$  for  $k \gg 0$ .

We denote the sheaf  $p_*\mathcal{F}(kE_\infty)$  by  $F_k$ . The  $T$ -Calogero-Moser space  $\mathfrak{CM}_n(E, T)$  is the moduli scheme of  $T$ -framed CM spectral sheaves  $(\mathcal{F}, \phi)$  for which the rank of the vector bundle  $W$  is  $n$ .

As we'll see (Definition 4.3, Lemma 4.4, and Corollary 4.12), the *spins* of generalized Calogero-Moser particles take value in the sheaf  $T$ . We will identify the moduli space  $\mathfrak{CM}_n^k(E) := \mathfrak{CM}_n(E, \mathcal{O}_b^k)$  with the completed phase space of the usual  $k$ -spin  $n$ -particle Calogero-Moser system. For general framing,  $\mathfrak{CM}_n(E, T)$  is identified with the framed Hitchin space  $\mathfrak{H}\mathfrak{CM}_n(E, T)$ . At the other extreme from the spin CM case  $T = \mathcal{O}_b^k$  we have the case of *simple framing*,  $T = \bigoplus_{i=1}^k \mathcal{O}_{x_i}$  with the  $x_i$  all distinct, which also generalizes the spinless case  $T = \mathcal{O}_b$ .

*Remark 4.7* (Normalization Conditions). The normalization conditions (i) and (ii) are open conditions on coherent sheaves of pure dimension one (and in fact any  $T$ -framed  $\mathcal{F}$  satisfying (i) must have pure dimension one). Condition (i) on the vector bundle  $W$  is discussed in Section 3.2: such  $W$  encode (via the Fourier transform) the positions of the Calogero-Moser particles. Condition (ii) is a normalization on the Hilbert polynomial of  $\mathcal{F}$  and should be considered part of the framing data. Note that (i) contains the case  $k = -1$  of (ii). In fact, it is easy to see that (i) and (ii) together imply Condition (ii) for all  $k$ : we use the exact sequence

$$0 \rightarrow F_{k-1} \rightarrow F_k \rightarrow T$$

to conclude that  $\deg(F_k) \leq \deg(F_{k-1}) + \deg(T)$ . Then induction on  $k$  gives that  $\deg(F_k) \leq (k+1)\deg(T)$  provided  $\deg(F_{-1}) = 0$ , with equality if and only if the above sequences are right exact for all  $k \geq 0$ .

It is also worth noting that if  $\mathcal{F}$  satisfies hypothesis (i) of the definition, then it satisfies hypothesis (ii) if and only if the natural map

$$p_*\mathcal{F}(kE_\infty) \rightarrow p_*(\mathcal{F}(kE_\infty)|_{E_\infty}) \cong T$$

is surjective for all  $k \geq 0$ .

*Remark 4.8* (Hamiltonians). The space of framed CM hamiltonians admit a natural geometric description on the moduli space of  $T$ -framed CM spectral sheaves. Indeed, following [DM, Hu, T, TV1, TV2], one may take a spectral sheaf to its scheme-theoretic support, viewed as a divisor on  $\overline{E}^\natural$  in the linear series

$$\mathbb{P} = |\mathrm{rk}(W) \cdot E_\infty + \deg(T) \cdot F|,$$

where  $F$  is a fiber of the projection  $\overline{E}^\natural \rightarrow E$ . The collection of divisors that contain the curve  $E_\infty$  with nonzero multiplicity form a hyperplane in  $\mathbb{P}$  with complement an affine space  $\mathbb{A}$ , and we obtain a natural map  $\mathfrak{CM}_n(E, T) \rightarrow \mathbb{A}$ . The framed CM Hamiltonians then come by pulling back a particular list of polynomials from  $\mathbb{A}$ .

**4.4. From Spectral Sheaves to Higgs Bundles.** In this section we identify the moduli spaces of framed spectral sheaves on  $\overline{E}^{\natural}$  and framed twisted Higgs bundles. This result is based on a “Koszul dual” description of framed sheaves due to L. Katzarkov, D. Orlov and T. Pantev [KOP].

**Theorem 4.9** ([KOP]). *There is a canonical equivalence between the category of  $T$ -framed CM spectral sheaves and that of Koszul data: quintuples  $(W, W', \iota, s, a)$  consisting of*

- (1)  $W \in \text{Bun}_n^{ss}(E)$ ,
- (2) an extension

$$0 \rightarrow W \xrightarrow{\iota} W' \xrightarrow{s} T \rightarrow 0$$

of  $W$  by  $T$ , and

- (3) a map  $a : A \otimes W \rightarrow W'$  extending  $\iota$  on  $W = \mathcal{O}_E \otimes W \subset A \otimes W$ .

**Construction 4.10.** The assignment of Koszul data to a spectral sheaf proceeds as follows. Let  $(\mathcal{F}, \phi)$  be a  $T$ -framed CM spectral sheaf. We set  $W = p_* \mathcal{F}(-E_\infty)$  and  $W' = p_* \mathcal{F}$ . Then  $W$  satisfies Condition (i) of Definition 4.6. Moreover, by Remark 4.7, the natural sequence

$$0 \rightarrow W \xrightarrow{\iota} W' \xrightarrow{s} T \rightarrow 0$$

is exact. Making the identification  $A = p_* \mathcal{O}(E_\infty)$ , we let  $a : A \otimes W \rightarrow W'$  be the restriction of the action of  $\mathcal{A}$  on sections of  $\mathcal{F}$ .

**Theorem 4.11.** *There is an isomorphism  $\mathfrak{CM}_n(E, T) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E, T)$  between the moduli of  $T$ -framed spectral sheaves and (untwisted) Higgs fields.*

*Proof.* We need to establish an equivalence between  $T$ -framed twisted Higgs bundles  $(W, \tilde{\eta}, u, v)$  and Koszul data  $(W, W', \iota, s, a)$  as in Theorem 4.9.

We first establish a bijection between the two types of data  $(W, W', \iota, s)$  and  $(W, u)$  coming from Koszul data and Higgs data, respectively. Given  $(W, W', \iota, s)$ , there is a natural map  $\tilde{u}$ ,

$$\begin{array}{ccccc} & & \tilde{u} & & \\ & \curvearrowright & & \curvearrowleft & \\ W' & \twoheadrightarrow & W'/\text{tors}(W') & \longrightarrow & W(S), \end{array}$$

induced by the isomorphism  $W|_{E \setminus S} \cong W'|_{E \setminus S}$ . The map  $\tilde{u}$  restricts to the identity on  $W$ , and we denote the associated quotient map  $T \xrightarrow{s^{-1}} W'/W \xrightarrow{\tilde{u}} W(S)/W$  by  $u$ ; we may also use the canonical identification  $W(S)/W = W/W(-S) = W|_S$  coming from the invariant differential of  $E$  to identify  $u$  with a map  $T \rightarrow W/W(-S) = W|_S$ .

Conversely, a diagram chase shows that  $W'$  is obtained (up to unique isomorphism) as the pullback of the exact sequence

$$0 \rightarrow W \rightarrow W(S) \rightarrow W(S)/W \rightarrow 0$$

along the map  $u : T \rightarrow W(S)/W$ . It is immediate that these two constructions give the bijections

$$(4.2) \quad \{(W, W', \iota, s)\} \rightleftharpoons \{(W, u)\}.$$



We thus obtain a diagram

$$(4.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & W & \xrightarrow{\iota} & W' & \xrightarrow{s} & T \longrightarrow 0 \\ & & \parallel & & \downarrow & \square & \downarrow u \\ 0 & \longrightarrow & W & \longrightarrow & W(S) & \longrightarrow & W(S)/W \longrightarrow 0, \end{array}$$

where the square marked  $\square$  is Cartesian, relating the corresponding data  $(W, W', \iota, s)$  and  $(W, u)$ .

It is now immediate from the universal property of pullbacks applied to (4.3) that there is a bijection between:

- (1) the set of maps  $a : A \otimes W \rightarrow W'$  such that  $a|_{\mathcal{O} \otimes W}$  is the identity on  $W$ .
- (2) The set of pairs

$$(A \otimes W \xrightarrow{\tilde{\eta}} W(S), A \otimes W \xrightarrow{\tilde{v}} T)$$

such that

- (a)  $\tilde{\eta}|_{\mathcal{O} \otimes W}$  is the identity on  $W$ ,
- (b)  $\tilde{v}|_{\mathcal{O} \otimes W} = 0$ , and
- (c) the diagram

$$\begin{array}{ccc} A \otimes W & \xrightarrow{\tilde{v}} & T \\ \downarrow \tilde{\eta} & & \downarrow u \\ W(S) & \longrightarrow & W(S)/W \end{array}$$

commutes.

Since the maps  $\tilde{v}$  in (2) are completely determined by the induced maps

$$W|_S = (A \otimes W / \mathcal{O} \otimes W)|_S \xrightarrow{v} T,$$

we find that the bijections of (4.2) extend to bijections

$$\{(W, W', \iota, s, a)\} \cong \{(W, \tilde{\eta}, u, v)\},$$

as desired. The proof of the functoriality properties of this bijection necessary to obtain a moduli isomorphism is straightforward, and we omit it.  $\square$

**Corollary 4.12.** *The moduli space of  $\mathcal{O}_b^k$ -framed spectral sheaves is a completed phase space for the  $k$ -spin Calogero-Moser system.*

*Proof.* This is immediate from Theorem 4.11 and Proposition 3.6.  $\square$

## 5. FLOWS ON SPECTRAL SHEAVES

**5.1. Tweaking Sheaves.** In this section we consider some variants of the simplest method of deforming sheaves on any variety  $Y$ , namely tensoring them by line bundles. If  $Y$  is a smooth projective variety, this gives rise to an action of the Picard group of  $Y$  on moduli spaces of sheaves on  $Y$ . In particular, the tangent space  $H^1(Y, \mathcal{O})$  to  $\text{Pic } Y$  at the trivial bundle gives rise to infinitesimal deformations of any sheaf. This infinitesimal action is defined for an arbitrary variety  $Y$  as the canonical map

$$H^1(Y, \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F})$$

for any sheaf  $\mathcal{F}$ . Concretely, a self-ext or first-order deformation of  $\mathcal{O}$  defines, via tensor product, a first-order deformation of any sheaf  $\mathcal{F}$ .

To construct particular deformations of sheaves (on reasonable varieties  $Y$ ), we can produce elements of  $H^1(Y, \mathcal{O})$  from local  $H^1$  of  $\mathcal{O}$  along divisors, or from arbitrary meromorphic functions on  $Y$ . If  $\mathcal{F}$  is a vector bundle, we may interpret the infinitesimal action of a meromorphic function  $f$  deforming  $\mathcal{F}$  as changing the transition functions of  $\mathcal{F}$  by scalar multiplication by  $f$  on the locus where  $f$  is defined. We may also work with a formal variant, deforming sheaves using Laurent series along a divisor in a smooth variety. For example, if  $X$  is a curve,  $x \in X$  and  $\mathcal{K}_x \supset \mathcal{O}_x$  are Laurent and Taylor series at  $x$  we have surjections

$$\mathcal{K}_x \twoheadrightarrow \mathcal{K}_x/\mathcal{O}_x = H_x^1(X, \mathcal{O}_X) \twoheadrightarrow H^1(X, \mathcal{O}_X)$$

from  $\mathcal{K}_x$  to local to global cohomology of  $\mathcal{O}$ , which we can use to construct deformations of sheaves.

More generally, take a local section of the local cohomology sheaf  $H_D^1(Y, \mathcal{O})$ , where  $D \subset Y$  is a divisor (locally principal subscheme). This section corresponds to an element of

$$\mathrm{Hom}(\mathcal{O}, \mathcal{O}(\infty D)/\mathcal{O}) = \mathrm{Hom}(\mathcal{O}, \widehat{\mathcal{O}}(\infty D)/\widehat{\mathcal{O}})$$

where we pass to completions along  $D$ . Tensoring this homomorphism by a sheaf  $\mathcal{F}$  we obtain a homomorphism

$$\mathcal{F} \rightarrow \widehat{\mathcal{F}}(\infty D)/\widehat{\mathcal{F}}.$$

Now we pull back the canonical extension

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}(\infty D) \rightarrow \widehat{\mathcal{F}}(\infty D)/\widehat{\mathcal{F}} \rightarrow 0$$

along this map and to obtain the desired local extension of  $\mathcal{F}$  by itself.

Thus we can deform sheaves on a curve using principal parts of functions at a point. We refer to this construction as “tweaking” of sheaves. The flows of many algebraically integrable systems can be described in this fashion (see e.g. [DM] where the Heisenberg flows of the KP hierarchies are described in this way and [BF1] for the case of generalized Drinfeld-Sokolov hierarchies). In the next section we explain how the flows of the  $GL_n$  Hitchin system, which are generically given by the action of Picard groups of spectral curves which live in the cotangent bundle  $T^*X$  of a curve, are in fact uniformly given by the action of the global cohomology  $H^1(T^*X, \mathcal{O})$  of the cotangent bundle.

**5.1.1. Tweaking Algebroids.** In this section we consider a more general construction, constructing arbitrary deformations of sheaves near a divisor  $D \subset Y$ . This is modeled on the loop algebra uniformization of moduli of bundles on a curve. The resulting flows do not commute in general and form an action of a Lie algebroid on moduli spaces of sheaves, rather than of a fixed Lie algebra (in other words, the space parameterizing deformations depends on the sheaf being deformed).

Let  $\mathcal{E}$  denote the  $\mathcal{O}_Y$ -algebra of Laurent series along the divisor  $D$ , i.e. functions on the punctured formal neighborhood of  $D$ . More precisely,  $\mathcal{E}$  is the inductive limit

$$\mathcal{E} = \varinjlim \widehat{\mathcal{O}}_{Y,D}(kD),$$

where  $\widehat{\mathcal{O}}_{Y,D}$  is the completion of  $\mathcal{O}_Y$  along  $D$ .

Consider a pair  $(\mathcal{F}, \xi)$  consisting of a coherent sheaf  $\mathcal{F}$  on  $Y$  and a Laurent endomorphism  $\xi \in \text{End}_{\mathcal{E}}(\mathcal{F}_{\mathcal{E}})$  (where  $\mathcal{F}_{\mathcal{E}} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$ ) of the restriction of  $\mathcal{F}$  to the punctured neighborhood of  $D$ . We then construct a first-order deformation of  $\mathcal{F}$ ,  $[\xi] \in \text{Ext}^1(\mathcal{F}, \mathcal{F})$ , as the image (under a connecting homomorphism) of the operation of restricting sections of  $\mathcal{F}$  to  $\mathcal{F}_{\mathcal{E}}$ :

$$\{s \mapsto \xi(s_{\mathcal{E}}) \bmod \mathcal{F}\} \in \text{Hom}(\mathcal{F}, \mathcal{F}_{\mathcal{E}}/\mathcal{F}) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}).$$

More geometrically (and informally), the deformation is defined by changing the transition function between the restrictions  $\mathcal{F}|_{Y \setminus D}, \mathcal{F}_{\hat{\mathcal{O}}_{Y,D}}$  of  $\mathcal{F}$  to  $Y \setminus D$  and to the completion along  $D$ : we define a new sheaf by multiplying the isomorphism between the restrictions of the two sheaves to the punctured formal neighborhood (which are both  $\mathcal{F}_{\mathcal{E}}$ ) by  $1 + \epsilon \cdot \xi$  over the dual numbers  $\mathbf{C}[\epsilon]/\epsilon^2$ .

In the case of vector bundles on a curve  $X$  with  $D = x \in X$ , this formal deformation procedure becomes the action of twisted loop algebras at  $x$  on the moduli space of vector bundles on  $X$ . If we trivialize a vector bundle  $\mathcal{F}$  near  $x$  then  $\xi$  becomes an element of the loop algebra  $L\mathfrak{gl}_n = \mathfrak{gl}_n \otimes \mathcal{E}$  of Laurent series of matrices at  $x$ , which acts on the moduli of bundles by infinitesimally changing the transition functions at  $x$ . Without the choice of trivialization, the twisted loop algebras  $\text{End}_{\mathcal{E}}(\mathcal{F}_{\mathcal{E}})$  form a transitive Lie algebroid over the moduli stack of bundles.

Similarly in the general setting we may consider all tweakings  $\xi \in \text{End}_{\mathcal{E}}(\mathcal{F}_{\mathcal{E}})$  as forming a Lie algebroid over the moduli stack  $\mathfrak{M}$  of coherent sheaves  $\mathcal{F}$  on  $Y$ . The tangent sheaf to  $\mathfrak{M}$  is the sheaf  $\underline{\text{Ext}}^1$  of self-extensions along  $Y$  of the universal sheaf  $\underline{\mathcal{F}}$  on  $\mathfrak{M}$ .

**Definition 5.1.**

- (1) The *algebroid of tweakings along  $D \subset Y$*  is the sheaf  $\underline{\text{End}}_{\mathcal{E}} = \text{End}_{\mathcal{E}}(\underline{\mathcal{F}}_{\mathcal{E}})$  on  $\mathfrak{M}$  of  $\mathcal{E}$ -module endomorphisms of the universal sheaf. The anchor map is the map  $\underline{\text{End}}_{\mathcal{E}} \rightarrow \underline{\text{Ext}}^1$  defined above.
- (2) The *algebroid of central tweakings along  $D$*  is the image of  $\mathcal{E}$  in  $\underline{\text{End}}_{\mathcal{E}}$ .

The central tweakings are the multiplication operators by functions on the support of a sheaf  $\mathcal{F}_{\mathcal{E}}$ . Note that if we consider sheaves where  $\mathcal{F}_{\mathcal{E}}$  is a line bundle on its support, then all tweakings are central:  $\text{End}_{\mathcal{E}}(\mathcal{F}_{\mathcal{E}})$  is given by functions  $\mathcal{E}|_{\text{Supp } \mathcal{F}_{\mathcal{E}}}$  on the support of  $\mathcal{F}_{\mathcal{E}}$ . Thus, in this generic case the Lie algebroid reduces to the commutative tweaking action of meromorphic functions considered above. This is, in particular, the case for CM spectral sheaves with simple framing, i.e.,  $T = \bigoplus_1^k \mathcal{O}_{x_i}$  is a direct sum of skyscrapers at distinct points of  $E_{\infty}$  (for example in the spinless case  $T = \mathcal{O}_b$ ). For general spectral sheaves, we obtain instead a richer nonabelian hierarchy of flows given locally by the action of several copies of  $L\mathfrak{gl}_k$  for different  $k$ .

**5.1.2. Twaking Framed Sheaves.** It is useful to consider also a relative version of the above constructions for sheaves framed along a divisor  $D$ . Namely, we would like to deform sheaves with a fixed restriction to  $D$ . First, we have an action of the group-scheme of line bundles equipped with a trivialization along  $D$ , or on the infinitesimal level of  $H^1(Y, \mathcal{O}(-D))$ , lifting the action of  $H^1(Y, \mathcal{O})$  on underlying unframed sheaves. More generally, meromorphic germs of functions along  $D$  act on framed sheaves in the same way as on unframed sheaves (in the latter case the action depends on the germ up to germs regular on  $D$ , while in the former germs are taken up to those vanishing along  $D$ ). More precisely, consider a coherent

sheaf  $T$  on  $D$  and the moduli stack  $\mathfrak{M}(T)$  of coherent sheaves  $\mathcal{F}$  on  $Y$  with an isomorphism  $\mathcal{F}|_D \rightarrow T$ . We have a forgetful map  $\mathfrak{M}(T) \rightarrow \mathfrak{M}$  to the moduli stack of the underlying unframed sheaves. We then have the following obvious lifting of the tweaking algebroid:

**Lemma 5.2.** *The pullback of the sheaf  $\underline{\text{End}}_{\mathcal{E}}$  of tweakings along  $D$  from  $\mathfrak{M}$  to  $\mathfrak{M}(T)$  has a canonical structure of Lie algebroid lifting the action on  $\mathfrak{M}$ . The anchor map on  $\mathfrak{M}$  vanishes on endomorphisms regular on  $D$ , while that on  $\mathfrak{M}(T)$  vanishes on endomorphisms vanishing on  $D$ .*

**5.2. Flows of Hitchin Systems.** In this section we give an explicit description of the correspondence between the Hitchin hamiltonians and their flows on moduli spaces of Higgs bundles. This description may be viewed as a geometric (or spectral) reformulation of the Lax pairs with spectral parameter for hamiltonian flows on loop algebras (see [AHP, BBT, DM, LM1, LM2]). The technique is based on the loop group uniformization of moduli of bundles, and parallels the discussion of isomonodromy flows in [BF2].

We start with a trivial statement about the prototype for the construction, the basic hamiltonian system on  $T^*GL_n$  (the geodesic flow on  $GL_n$ ). Let us identify  $\mathfrak{gl}_n^*$  with  $\mathfrak{gl}_n$  by the trace form and  $T^*GL_n$  with  $GL_n \times \mathfrak{gl}_n$  by left translation. For every positive integer  $i$ , we have a function on  $T^*GL_n$  given by  $\frac{1}{i+1} \text{Tr } Y^{i+1}$  on the  $\mathfrak{gl}_n$  component. There are two natural ways to write the corresponding hamiltonian vector field. In Lax form, we describe the vector field at  $(X, Y) \in GL_n \times \mathfrak{gl}_n$  as  $([Y^i, \cdot], 0)$ . On the other hand we have a spectral interpretation of this flow. We will identify the adjoint quotient stack  $\mathfrak{gl}_n/GL_n$  with the space of torsion coherent sheaves on the affine line  $\text{Spec } \mathbf{C}[t]$  of length  $n$ . The Lie algebra  $\mathfrak{gl}_n$  itself is identified with  $\mathbf{C}[z]$  module structures on  $\mathbf{C}^n$ , i.e. torsion sheaves as above with a basis. Finally  $GL_n \times \mathfrak{gl}_n$  is identified with torsion sheaves with two bases, in other words module structures on  $\mathbf{C}^n$  plus an additional basis for  $\mathbf{C}^n$ . In this formulation, the hamiltonian vector field above simply rescales the second basis by the action of the function  $z^i$  on  $\text{Spec } \mathbf{C}[z]$ .

We would like to have an analogous spectral description of Hitchin hamiltonian flows, by reinterpreting the corresponding Lax form. Let  $X$  denote a smooth connected projective curve, and  $Bun_n(X)$  the moduli stack of rank  $n$  vector bundles on  $X$ . Fix a point  $x \in X$ , and let  $\mathcal{K} \simeq \mathbf{C}((z))$  and  $\mathcal{O} \simeq \mathbf{C}[[z]]$  denote the complete local field and ring at  $x$  (functions on the punctured disc  $D^\times$  and disc  $D$  at  $x$ , respectively). We denote by  $Bun_n(X, x)$  the moduli scheme of bundles with a trivialization on  $D$ , i.e. an infinite level structure at  $x$ . The cotangent bundles  $T^*Bun_n(X)$  and  $T^*Bun_n(X, x)$  are identified with the moduli of rank  $n$  Higgs bundles  $\eta \in H^0(X, \text{End } V \otimes \Omega)$  ( $V \in Bun_n(X)$ ) on  $X$  and the moduli of Higgs bundles  $\eta \in H^0(X \setminus x, \text{End } E \otimes \Omega)$  ( $V \in Bun_n(X, x)$ ) with arbitrary pole at  $x$ , respectively.

We will identify these cotangent bundles with moduli spaces of spectral sheaves on  $T^*X$  [Hi, M, DM]—namely a Higgs field  $\eta$  on  $V \in Bun_n(X)$  gives  $V$  an  $\mathcal{O}_{T^*X}$ -module structure, defining a coherent sheaf on  $T^*X$  which pushes forward to  $V$ . Likewise a meromorphic Higgs field  $\eta$  as above makes  $V(\infty \cdot x) = j_* j^* V$  (where  $j$  is the inclusion of  $X \setminus x$  into  $X$ ) into an  $\mathcal{O}_{T^*X}$ -module. The support of this sheaf (the *spectral curve*) is the zero locus of the characteristic polynomial of  $\eta$ , i.e. the spectral curve is  $X_\eta = \text{Spec}_X \mathcal{O}_{T^*X} / \{\text{char } \eta\}$ . We will describe the Hitchin flows on this data in the same way as in the prototype example (with the Higgs field as an  $\Omega$ -twisted version of the matrix  $Y$  over the punctured disc).

The spectral sheaf interpretation of the Hitchin system allows us to define infinitesimal actions of  $H^1(T^*X, \mathcal{O})$  on  $T^*Bun_n(X)$  and of meromorphic germs along the cotangent fiber  $F_x \subset T^*X$  on  $T^*Bun_n(X, x)$ . The latter is defined by tweaking the corresponding spectral sheaf as before, and preserving the trivialization of  $V|_D$  (note that this is well-defined since the tweaking flows canonically preserve  $V|_D$  while changing the gluing with  $V|_{X \setminus x}$ ).

Let

$$\begin{aligned} \text{Hitchin}(X) &= \bigoplus_{n=1}^{\infty} H^0(X, \Omega^{i+1}) \subset \\ \text{Hitchin}(X, x) &= \bigoplus_{i=0} H^0(X \setminus x, \Omega^{\otimes i+1}) \subset \\ \text{Hitchin}(D^\times) &= \bigoplus_{i=0} \Omega_{\mathcal{K}}^{i+1} \end{aligned}$$

be the infinite Hitchin base spaces (where  $\Omega_{\mathcal{K}} \simeq \mathbf{C}((z))dz$  is the space of Laurent differentials at  $x$ ). The Hitchin maps

$$H : T^*Bun_n(X) \rightarrow \text{Hitchin}(X), \quad H : T^*Bun_n(X, x) \rightarrow \text{Hitchin}(X, x)$$

have as the  $i$ th component

$$H_i : (V, \eta) \mapsto \frac{1}{i+1} \text{Tr}(\eta^{i+1}).$$

Note that we write the Hitchin map in the basis for invariant polynomials coming from traces of powers, rather than the usual basis consisting of coefficients of the characteristic polynomials.

The topological dual of the (Tate) vector space  $\mathcal{K}$  is identified by the residue pairing with the differentials  $\Omega_{\mathcal{K}}$ . Likewise if  $\mathcal{T}_{\mathcal{K}} \simeq \mathbf{C}((z))\partial_z$  is the space of Laurent vector fields, then  $\mathcal{T}_{\mathcal{K}}^i$  is identified with the dual to  $\Omega_{\mathcal{K}}^{i+1}$ . Let

$$\mathcal{O}(T^*D^\times) = \bigoplus \mathcal{T}_{\mathcal{K}}^i$$

denote functions on the (suitably defined) cotangent bundle of  $D^\times$ .

**Lemma 5.3.** *The graded dual spaces of the Hitchin spaces are canonically identified as follows:*

$$\begin{aligned} \text{Hitchin}(D^\times)^* &= \mathcal{O}(T^*D^\times) \twoheadrightarrow \\ \text{Hitchin}(X, x)^* &= \mathcal{O}(T^*D^\times) / \mathcal{O}(T^*(X \setminus x)) \twoheadrightarrow \\ \text{Hitchin}(X)^* &= H^1(T^*X, \mathcal{O}) \end{aligned}$$

It follows that we may identify  $H^1$  classes on  $T^*X$  with linear functions on the Hitchin base space, and meromorphic germs on  $T^*X$  along the fiber  $F_x$  with linear functions on the meromorphic Hitchin space.

**Theorem 5.4.** *For a class  $\xi \in H^1(T^*X, \mathcal{O})$  (respectively  $\xi \in \mathcal{O}(T^*D^\times)$ ), the hamiltonian vector field on  $T^*Bun_n(X)$  ( $T^*Bun_n(X, x)$ ) associated to  $H^*\xi$  ( $H_x^*\xi$ ) is identified with the tweaking action of  $\xi$ .*

*Proof.* Morally, the theorem follows by hamiltonian reduction from the corresponding statement on  $T^*GL_n(\mathcal{K})$ . More concretely, let  $(V, \eta) \in T^*Bun_n(X, x)$ . The tangent space to the moduli space at  $(V, \eta)$  is given by

$$(s, \theta) \in \text{End}_{D^\times}(V) / \text{End}_{X \setminus x}(V) \oplus H^0(X \setminus x, \text{End}(V) \otimes \Omega).$$

The symplectic form on the tangent space is given by the residue of the trace pairing:

$$\omega((s_1, \theta_1), (s_2, \theta_2)) = \text{Res}(\text{Tr}(s_1 \theta_2 - s_2 \theta_1)).$$

Fix  $\xi \in \mathcal{T}_{\mathcal{K}}^i \subset \mathcal{O}(T^*D^\times)$  (we assume  $\xi$  homogeneous for simplicity of notation). The corresponding Hitchin hamiltonian is  $H_\xi(V, \eta) = \text{Res}(\xi \text{Tr}(\frac{1}{i+1} \eta^{i+1}))$ . Perturbing  $(V, \eta)$  by  $(s, \theta)$  as above, we find the differential of this function is

$$dH_\xi|_{(V, \eta)}(s, \theta) = \text{Res}(\xi \text{Tr}(\eta^i \theta)).$$

To calculate the hamiltonian vector field  $v_\xi|_{(V, \eta)} = (s_\xi, \theta_\xi)$ , we must solve

$$\text{Res}(\xi \text{Tr}(\eta^i \theta)) = dH_\xi|_{(V, \eta)}(s, \theta) = \omega_{(V, \eta)}(v_\xi, (s, \theta)) = \text{Res}(\text{Tr}(s_\xi \theta)) - \text{Res}(\text{Tr}(s \theta_\xi)).$$

It is immediate that  $v_\xi|_{(V, \eta)} = (\xi \eta^i, 0)$ . In other words, we have written the Hitchin flows in Lax form, with the flow at  $(V, \eta)$  given by the action on  $V$  of the element  $\xi \eta^i \in \text{End}(V|_{D^\times})$ .

On the other hand the tweaking action of a homogeneous  $\xi \in \mathcal{T}_{\mathcal{K}}^i$  is given by multiplication by  $\xi \in \mathcal{O}(T^*D^\times)$  as an endomorphism of  $V|_{D^\times}$ , considered as an  $\mathcal{O}(T^*D^\times)$ -module using  $\eta|_{D^\times}$ . However this endomorphism is simply the product  $\xi \eta^i \in \text{End}_{D^\times}(V)$ , and the tweaking action by this endomorphism is precisely the Lax vector field we derived above. Thus the (central) tweaking flow by  $\xi$  on  $T^*X$  is written as an element of the tweaking algebroid at  $x$  on  $X$ , where it is identified with the loop algebra action on bundles  $V$  with trivialization on the disc.

Finally note that if  $\xi \in \mathcal{O}(T^*(X \setminus x)) \subset \mathcal{O}(T^*D^\times)$ , then on the one hand the Hitchin hamiltonian defined by  $\xi$  vanishes on  $T^*Bun_n(X)$ , while on the other the tweaking action of  $\xi$  on spectral sheaves is trivial, since it is given by a global rescaling on  $X \setminus x$  which vanishes in  $\text{Ext}^1$ . This establishes the theorem in the meromorphic case.

For  $(V, \eta) \in T^*Bun_n(X)$ , we choose a trivialization of  $V|_D$  to put ourselves in the previous situation. The Hitchin polynomials of  $\eta$  are now regular on  $D$ , so  $\xi \in \mathcal{O}(T^*D) \subset \mathcal{O}(T^*D^\times)$  pull back to the zero function of  $(V, \eta)$ . Likewise, the tweaking action of such  $\xi$  corresponds to a regular endomorphism of  $V$  on  $T^*D$ , which corresponds to a change of trivialization of  $V$  on  $D$ . Thus the action on  $T^*Bun_n(X)$  descends to

$$\mathcal{O}(T^*(X \setminus x)) \backslash \mathcal{O}(T^*D^\times) / \mathcal{O}(T^*D) = H^1(T^*X, \mathcal{O}),$$

as desired. □

*Remark 5.5.* For simplicity, we have stated the theorem in the setting of a smooth curve  $X$ . However, the result carries over to a more general setting. In particular we are interested in the case when  $X = E$  is a cubic curve,  $x \in \mathbf{G}$  is a smooth point and  $T^*E$  is the log cotangent bundle. It is evident that the above arguments (which are local at  $x$ ) extend automatically to this setting.

**5.3. Compatibility of Hitchin and Heisenberg Flows.** In [DM] and [LM2], the authors investigate the compatibility of the Hitchin systems on  $T^*Bun_n(X, D)$  with KP- or KdV-type flows on moduli of spectral curves defined using the Krichever construction. More precisely, if we fix a point  $x \in X$  and a partition  $p$  of  $n$ , we can look at the moduli stack  $\text{Higgs}_{x, p} \rightarrow T^*Bun_n(X, D)$  of Higgs bundles whose spectral curve over the punctured neighborhood of  $x$  is identified with a fixed spectral cover  $\Sigma_p \rightarrow D^\times$  over the punctured disc, with ramification type  $p$ . The partitions  $p$

label conjugacy classes of Heisenberg (or Cartan) subalgebras of the loop algebra  $L\mathfrak{gl}_n$ , consisting of loops into diagonalizable matrices whose eigenvalues undergo a permutation of type  $p$  around the puncture. By fixing a distinguished ramified cover  $\Sigma_p$  of the punctured disc of type  $p$ , we pick out a particular Heisenberg algebra  $A_p \subset L\mathfrak{gl}_n$ , isomorphic to functions on  $\Sigma_p$ , and there is a natural action of  $H_p$  on  $\text{Higgs}_{x,p}$  by tweakings, modifying the Higgs bundle but preserving its spectral curve (see [AB] and [BF1] for detailed discussions of Heisenberg and KdV flows). The above papers prove that the actions of these Heisenberg algebras are hamiltonian with respect to the Poisson structure on Higgs bundles, when sufficiently strong conditions are imposed on the regularity of the spectral curves or the Higgs fields at  $x$ . Since the  $A_p$  action is given by tweakings, we immediately recover from Theorem 5.4 a strong form of the compatibility, independent on the singularities of the spectral curve. (Note that on  $\text{Higgs}_{x,p}$  all spectral sheaves are rank one on their support near  $x$ , so all tweakings along  $F_x$  are central.)

**Corollary 5.6** (Compatibility). *Fix  $x \in X$  and a partition  $p$  of  $n$ . The algebroid of tweakings along  $F_x \subset T^*X$  pulls back to the action of the Heisenberg algebra  $A_p$  on  $\text{Higgs}_{x,p}$ . Therefore all Heisenberg flows are given by pullbacks of Hitchin hamiltonians.*

**5.4. Framed Calogero-Moser Hierarchies.** In this section we consider the tweaking flows on CM spectral sheaves. Since we are considering sheaves on  $\overline{E}^\natural$  framed along  $E_\infty$ , a natural collection of tweaking flows is parametrized by the algebroid of tweakings along  $E_\infty$ :

**Definition 5.7.** The CM algebroid on  $\mathfrak{CM}_n(E, T)$  is the Lie algebroid  $\underline{\text{End}}_{\mathcal{E}}$  of tweakings along the divisor  $E_\infty \subset \overline{E}^\natural$ .

We would like to obtain concrete hierarchies of tweakings, specifically tweaking by meromorphic functions (central tweakings). To do so we single out particular meromorphic germs on  $\overline{E}^\natural$  by which to tweak. We will then identify the resulting flows with Hitchin hamiltonian flows on Higgs bundles.

Let us consider first the  $k$ -spin case  $T = \mathcal{O}_b^k$ . Recall the meromorphic function  $\underline{t}$  on  $E^\natural$  defined near the fiber  $F_b$  over  $b$  (Section 4.1). We will define flows on CM spectral sheaves as tweaking near  $b \in E_\infty$  by powers of  $\underline{t}$ . Note that the definition of  $\underline{t}$  required a choice of global vector field on  $E$ , equivalently a trivialization of the cotangent fiber of  $E$ . Accounting for the action of change of trivialization, we identify the resulting polynomial algebra  $\mathbf{C}[\underline{t}]$  canonically with  $\mathbf{C}[\partial]$  where  $\partial$  denotes a global vector field on  $E$ .

Given a polynomial  $p$  in  $\mathbf{C}[\underline{t}]$  (considered as a germ of a meromorphic function near the fiber  $F_b$ ) and a CM spectral sheaf, we may restrict  $p$  to define a germ of a meromorphic function on the corresponding CM spectral curve at its intersection point  $b_\infty$  with  $E_\infty$ . This defines an  $H^1$  class of  $\mathcal{O}(-E_\infty)$ , which then acts as a deformation of the framed spectral sheaf. Thus we have an action of  $\mathbf{C}[\underline{t}]$  by commuting vector fields on  $\mathfrak{CM}_n^k(E)$ :

**Definition 5.8** (CM flows). The *spin CM hierarchy* on spectral sheaves is the canonical action of  $\mathbf{C}[\partial]$  on  $\mathfrak{CM}_n^k(E)$ , where  $\partial^i$  acts by deforming spectral sheaves by the restriction of  $\underline{t}^i$ .

We may similarly define CM flows on  $T$ -framed spectral sheaves for an arbitrary  $T$ . Let  $S = \coprod_{j=1}^l x_j \subset E$  denote the set-theoretic support of  $T$ , consisting of  $l$  distinct points on  $X$  (counted without multiplicity).

Let  $\widehat{F}_j$  denote the formal neighborhood of the fiber  $F_{x_j} \subset E^\natural$ . For  $x_j \neq b$ , we consider the restriction of the polynomial ring  $\mathbf{C}[\partial]$  of global functions on  $T^*E$  to  $\widehat{F}_j$ , giving meromorphic endomorphisms of framed spectral sheaves. When  $x_j = b$ , we shift these functions by the zeta function as above, again giving an action of  $\mathbf{C}[\partial]$  (as powers of  $\underline{t}$ ) as meromorphic endomorphisms of framed spectral sheaves restricted to  $\widehat{F}_b$ .

**Definition 5.9.** The *framed CM hierarchy* on  $\mathfrak{CM}_n(E, T)$  is the action of  $\mathbf{C}[\partial]^l$  on  $\mathfrak{CM}_n(E, T)$ , where  $\partial^i$  in the  $j$ th copy acts by tweaking on  $\widehat{F}_j$  by  $\partial^i$  for  $x_j \neq b$  and by  $\underline{t}^i$  for  $x_j = b$ .

**Theorem 5.10.**

- (1) The identification  $\mathfrak{CM}_n(E, T) \rightarrow \mathfrak{H}\mathfrak{CM}_n(E, T)$  of Theorem 4.11 (in particular  $\mathfrak{CM}_n^k(E) \rightarrow \mathfrak{H}\mathfrak{CM}_n^k(E)$ ) intertwines the framed CM hierarchy (action of  $\mathbf{C}[\partial]^l$ ) with the flows of the framed CM hamiltonians, Definition 4.5:  $\partial^i$  at the point  $x_j$  is the hamiltonian flow for  $H_{i, x_j}$ .
- (2) For simple framing  $T = \bigoplus_{i=1}^k \mathcal{O}_{x_i}$  this construction gives all tweaking flows: the action of the full CM algebroid is identified with the action of  $\mathbf{C}[\partial]^k$ .

*Proof.* By Theorem 5.4, the Hitchin flows are defined by tweaking of spectral sheaves on  $T^*E$ , while the CM spectral flows are defined by tweaking sheaves on  $E^\natural$  framed along  $E_\infty$ . More specifically, the hamiltonian flow of the spin CM hamiltonian  $H_{i, b}$ , given by a residue of a trace of  $(\eta + \zeta \text{Id})^i$ , is identified by Theorem 5.4 with the tweaking action of the function  $(\partial + \zeta)^i$  on  $T^*E$ , which has a pole along the fiber  $F_b$  over  $b$ . This function is identified with  $\underline{t}^i$  by the birational identification of  $T^*E$  and  $E^\natural$ . It follows that under the isomorphism of Corollary 4.11, the meromorphic endomorphism of a Higgs bundle given by  $(\partial + \zeta)^i$  is identified with the meromorphic endomorphism of a twisted Higgs bundle, hence CM spectral sheaf, given by  $\underline{t}^i$ . For  $x_j \neq b$ , our hamiltonians are simply residues of traces of powers of the Higgs field, which correspond to multiplication by powers of the coordinate function  $\partial$  on the fibers of  $T^*E$  (which is identified with  $E^\natural$  away from  $F_b$ ). (Note that the tweaking flows preserve the framing at  $E_\infty$ , while the Hitchin hamiltonians  $H_i$  are independent of the framing data  $u, v$ .) It follows that the corresponding tweaking flows are identified, as claimed.

Part (2) follows from the observation (Section 5.1.1) that for simple framing, the  $T$ -framed spectral sheaves  $\mathcal{F}$  are all rank one torsion-free sheaves on their support. Hence all tweakings along  $E_\infty$  are given by multiplication by meromorphic functions on the support of the microlocalization  $\mathcal{F}_\mathcal{E}$ , and considered up to functions that vanish on the intersection with  $E_\infty$ . Finally the support of  $\mathcal{F}_\mathcal{E}$  is a union of  $k$  punctured discs  $D_j^\times$  (one through each  $x_j \in E_\infty$ ). Hence functions on  $D_j^\times$  are identified (via restriction) with Laurent series  $\mathbf{C}((\partial^{-1}))$  in the fiber coordinate  $\partial$  on  $T^*E$  (or  $\underline{t}$  for  $x_j = b$ ). The assertion follows from the observation the polynomial algebra  $\mathbf{C}[\partial]$  maps isomorphically to the quotient of  $\mathbf{C}((\partial^{-1}))$  by functions vanishing at  $x_j$ . □



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